

JOURNAL OF APPROXIMATION THEORY 1, 1-4 (1968)

Derivatives of Polynomials with Positive Coefficients¹

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The paper [1] contains the following result. There exists a constant $C > 0$ such that for each polynomial of the form $P_n(x) = \sum_0^n b_k x^k (1-x)^{n-k}$, $b_k \geq 0$,

$$\|P_n'\|/\|P_n\| \leq Cn, \quad n = 1, 2, \dots, \quad (1)$$

for the uniform norm on $[0, 1]$. This relation can also be written

$$\frac{\|P_n'\|}{\|P_n\|} \leq C \frac{\|p_n'\|}{\|p_n\|}, \quad (2)$$

where p_n are the special polynomials $p_n(x) = x^n$.

In particular, (1) holds for polynomials with positive coefficients in x ,

$$P_n(x) = \sum_{k=0}^n a_k x^k, \quad a_k \geq 0.$$

This follows from (1), but also immediately, since with P_n , also P_n' is a polynomial with positive coefficients, and since for such P_n , $\|P_n\| = P_n(1)$. In the present note we prove the inequality (2) for the infinite interval $(0, +\infty)$, and for a supremum norm with weight. The norm of a function f on $(0, +\infty)$ is given by

$$\|f\| = \sup_{x \geq 0} |f(x) e^{-\omega(x)}|, \quad (3)$$

where ω increases on $(0, +\infty)$. In addition to some mild smoothness requirements for ω , we shall assume that ω does not increase too slowly. Thus, Theorem 2 allows $\omega(x) = \log^p x$, $p > 1$, but functions $\omega(x) = A \log x$ are, of course, excluded.

In what follows, we shall assume that $\omega(x)$ is a positive differentiable function, defined for $0 \leq x < +\infty$, increasing strictly to $+\infty$, and such that also $x\omega'(x)$ strictly increases to $+\infty$. For each $n = 0, 1, \dots$, the maximum of $x^n e^{-\omega(x)}$ is attained at a unique point $x = x_n$, given by

$$n = x_n \omega'(x_n). \quad (4)$$

¹ This work has been supported, in part, by the Contract No. AF49(638)-1401 of the Office of Scientific Research, U.S. Air Force.

We have $x_n \nearrow +\infty$ as $n \rightarrow \infty$. Another remark is that the behavior of $\|p_n'\|/\|p_n\|$ is very much like that of $n/x_n = \omega'(x_n)$:

$$\frac{n}{x_n} \leq \frac{\|p_n'\|}{\|p_n\|} \leq \frac{n}{x_{n-1}}, \quad n = 2, 3, \dots \quad (5)$$

This follows from the inequalities

$$\begin{aligned} \|p_n\| &= \frac{x_n}{n} n x_n^{n-1} e^{-\omega(x_n)} \leq \frac{x_n}{n} \|p_n'\|, \quad n \geq 1, \\ \|p_n'\| &= \frac{n}{x_{n-1}} x_{n-1}^n e^{-\omega(x_{n-1})} \leq \frac{n}{x_{n-1}} \|p_n\|, \quad n \geq 2. \end{aligned}$$

The following two theorems deal, roughly, with the cases when $\omega'(x)$ increases and when it decreases.

THEOREM 1. *Let $\omega(x)$ satisfy the inequalities $\omega(x) - \omega(0) \leq Ax\omega'(x)$, $x \geq 0$, and $\omega'(y) \leq A\omega'(x)$, $y \leq x$, for some constant $A > 0$. (Both conditions are satisfied if ω' increases or if ω' decreases, but remains bounded away from zero.) Then for some constant $C > 0$, inequality (2) holds for each polynomial P_n with positive coefficients.*

THEOREM 2. *Let $\overline{\lim}_{x \rightarrow \infty} \omega'(x) < 1$ and assume that for some $0 < q < 1$,*

$$\omega(x) \leq qx\omega'(x) \log \frac{1}{\omega'(x)} \text{ for all sufficiently large } x. \quad (6)$$

Then for some constant $C > 0$,

$$\frac{\|P_n'\|}{\|P_n\|} \leq C, \quad n = 1, 2, \dots \quad (7)$$

Note that if ω' is bounded, then according to (5), $\|p_n'\|/\|p_n\| \leq \text{Const}$. The proof depends upon the following

LEMMA. *Let $u_k = \frac{1}{2}(k+1)^{-2}$. There exists a $k_0 \geq 1$ for which*

$$\frac{x^k e^{-\omega(x)}}{\|p_k\|} < u_k, \quad k \geq k_0, \quad (8)$$

if x and k satisfy $x < k$ in case of Theorem 2 and $x < cx_k$ in case of Theorem 1 (where $c > 0$ is a constant).

Proof of the Lemma. Assume that (8) is violated for some k . Then

$$k \log(x/x_k) - \omega(x) + \omega(x_k) \geq \log u_k. \quad (9)$$

In case of Theorem 1, we have $\omega(x_k) - \omega(x) \leq Ax_k \omega'(x_k) = Ak$; hence (9) implies

$$k \log(x/x_k) \geq -Ak + \log u_k \geq -(A+1)k, \quad k \geq k_0,$$

hence $x/x_k \geq c$, $c = e^{-(A+1)}$.

Likewise, in the case of Theorem 2, there is, according to (6), a k_0 so that, for $k \geq k_0$,

$$\omega(x_k) \leq qk \log \frac{1}{\omega'(x_k)} < k \log \frac{1}{\omega'(x_k)} + \log u_k.$$

Therefore, for $k \geq k_0$, (9) implies

$$k \log(x/x_k) \geq k \log \omega'(x_k)$$

or

$$x \geq x_k \omega'(x_k) = c.$$

To complete the proof of the theorems, let $S = S(x, n)$ be the set of integers k which satisfy $k_0 < k \leq n$ and the inequality $cx_{k-1} \leq x$ (in case of Theorem 1) or $k-1 \leq x$ (in case of Theorem 2). Let L be the remaining integers k with $k_0 < k \leq n$.

For a polynomial with positive coefficients,

$$P_n(t) = \sum_{k=0}^n a_k t^k,$$

we put

$$Q_n(t) = \sum_{k_0 < k \leq n} a_k t^k.$$

We can assume that $\|Q_n\| > 0$, for if $Q_n(t)$ vanishes, the following proof is simplified. Let x be such that $Q_n'(x) = \|Q_n'\|$ (obviously, $x_1 \leq x \leq x_n$). Then

$$\|P_n'\| \leq \sum_{k \leq k_0} ka_k \|p_{k-1}\| + \|Q_n'\| = \Sigma_1 + \|Q_n'\|; \quad (10)$$

$$\|Q_n'\| = \sum_{k \in L} ka_k x^{k-1} e^{-\omega(x)} + \sum_{k \in S} = \Sigma_2 + \Sigma_3,$$

say. With $M = \max_{k \leq k_0} (k \|p_{k-1}\| / \|p_k\|)$,

$$\Sigma_1 \leq M \sum_{k \leq k_0} \|a_k p_k\| \leq (k_0 + 1) M \|P_n\|. \quad (11)$$

For $k \in L$, we have (8) with k replaced by $k-1$. Therefore

$$\Sigma_2 \leq \sum_{k_0 < k \leq n} ka_k \|p_{k-1}\| \frac{1}{2k^2} \leq \|Q_n'\| \sum_{k=1}^{\infty} \frac{1}{2k^2} = (1-a) \|Q_n'\|,$$

where a , $0 < a < 1$, is an absolute constant. This implies that

$$\Sigma_3 \geq a \|Q_n'\|. \quad (12)$$

Our last computation is different in the cases of Theorem 1 and Theorem 2. We note that

$$\|P_n\| \geq \sum_{k \in S} \frac{x}{k} k a_k x^{k-1} e^{-\omega(x)}.$$

In the first case, with $c_1 = ac$,

$$\|P_n\| \geq c \sum_{k \in S} \frac{x_{k-1}}{k} k a_k x^{k-1} e^{-\omega(x)} \geq c_1 \|Q_n'\| \min_{1 \leq k \leq n} \frac{1}{\omega'(x_k)}.$$

Hence, by the assumptions of Theorem 1 and (5),

$$\begin{aligned} \|Q_n'\| &\leq \frac{1}{c_1} \max_{1 \leq k \leq n} \omega'(x_k) \|P_n\| \\ &\leq \frac{A}{c_1} \omega'(x_n) \|P_n\| \leq \frac{A}{c_1} \|P_n\| \frac{\|p_n'\|}{\|p_n\|}. \end{aligned} \quad (13)$$

From (10), (11), and (13) we obtain (2), since $\|p_n'\|/\|p_n\|$ bounded from below.

In the second case, since $x/k > \frac{1}{2}$ for $x \in S$,

$$\|P_n\| \geq \frac{1}{2} \sum_{k \in S} k a_k x^{k-1} e^{-\omega(x)} = \frac{1}{2} \Sigma_3 \geq \frac{a}{2} \|Q_n'\|, \quad (14)$$

and we obtain (7) from (10), (11), and (14).

We make some additional remarks. In [3], Szegő studied the order of magnitude of $\|p_n'\|/\|p_n\|$ for unrestricted polynomials P_n , for the norm $\|f\| = \sup_{x \geq 0} |f(x) e^{-x}|$ on $(0, +\infty)$. He obtained that this does not exceed $C_0 n$. (For the Laguerre polynomials P_n , the quotient is $= n$.) In this case, the largest value of $\|p_n'\|/\|p_n\|$ is $= e$ for $n = 1$; this quotient decreases and has limit 1 for $n \rightarrow \infty$. We see that $\|p_n'\|/\|p_n\|$ is much smaller for polynomials with positive coefficients, than in the general case.

It has been found [2] that the smallest possible constant C in (1) is $C = e$. Of some interest is the smallest value of Szegő's constant C_0 ; this has not yet been determined. A possible conjecture is that this, too, is $C_0 = e$.

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